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ANALYSIS OF FINITE COMPOSITE LAMINATES WITH HOLES

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Abstract—In this study, the Modified Mapping Collocation method is employed to analyse the response of finite composite laminates with multiple circular holes under appropriate material and geometric symmetry conditions. In the case of a laminate with a single hole, the analysis is capable of accommodating the direct imposition of traction conditions, as well as the displacement constraints, without requiring any symmetry conditions. The results presented in this study are in agreement with existing solutions. The results also illustrate the effect of finite boundaries and their relative position in relation to the holes on the stress concentrations.

1. INTRODUCTION

The application of composite materials in primary aircraft structures has led to extensive research and development, much of which has been aimed at developing an understanding of the sensitivity of composite laminates to the presence of hole(s). In order to better understand the complex behavior of such structures, experimental investigations into the failure of composite laminates with holes are still continuing (Poon, 1991). Along with the experimental efforts, analytical modeling of the problem has attracted the attention of numerous researchers, especially in the case of holes subjected to bearing load due to fasteners.

Prior analytical investigations employed a variety of techniques, ranging from approximate solutions to comprehensive numerical methods. The finite element and boundary collocation methods are the numerical techniques most commonly used to obtain the stress distribution around a hole in a composite plate with finite geometry under tractions or displacement constraints. Although the finite element method is useful for calculating accurate results for a particular problem, it is not suitable for iterative design calculations for optimizing laminate construction in the presence of holes. The Modified Mapping Collocation (MMC) method (a variation of the boundary collocation method) introduced by Bowie and Neal (1970) was a better alternative and was adopted by Oplinger and Gandhi (1974), Ogonowski (1980) and Wilmarth (1982) to determine the stress field in a finite geometry orthotropic laminate with a hole.

These prior investigations have a significant limitation in that they are incapable of enforcing displacement constraints directly. Bowie (1974) recognized this shortcoming and avoided it by inverting the displacement constraint to a corresponding average traction along the boundary. This inversion results in the imposition of tractions and the treatment of the displacement constraint as an unknown. Recently, Cheong and Hong (1989) employed this approach to investigate the effect of grip conditions on an orthotropic finite plate containing a circular hole with edge cracks. Although this indirect way of imposing displacement constraints yields acceptable results, it limits the applicability of the MMC method to the investigation of various other problems involving symmetry and mixed boundary conditions.

Recently, Madenci and Ileri (1991) re-examined the MMC method and presented a procedure for directly applying the traction conditions, as well as the displacement constraints and combinations thereof, within the framework of plane problems of anisotropic elasticity. The analysis presented herein to investigate the response of composite laminates

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with multiple holes under the appropriate symmetry conditions is a result of this improvement.

A summary of the complex variable formulation and the treatment of the resulting integration constants is presented in Section 2, along with a description of the numerical solution method for direct application of various boundary conditions. The numerical results for a finite geometry composite laminate with a single hole, two holes, and four holes subjected to uniform loading at the edges are given in Section 3.

2. MATHEMATICAL FORMULATION AND SOLUTION

Under plane stress assumptions, Lekhnitskii (1968) developed a solution procedure with two analytic functions, $\Phi_1(z_1)$ and $\Phi_2(z_2)$, satisfying the equations of equilibrium and compatibility in Cartesian coordinates (x, y) for anisotropic laminates. The variables z_1 and z_2 are complex and are given by $z_1 = x+\mu_1 y$ and $z_2 = x+\mu_2 y$. The complex parameters μ_1 and μ_2 are the roots of the characteristic equation derived by Lekhnitskii,

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12}+a_{66})\mu^2 - 2a_{66}\mu + a_{22} = 0, \tag{1}$$

in which a_{ij} $(i, j = 1, 2, 6)$ are the compliance coefficients of a laminate.

The stress and displacement components can be expressed as

$$\begin{aligned}
\sigma_{xx} &= 2\,\mathscr{Re}\,[\mu_1^2\Phi_1'(z_1)+\mu_2^2\Phi_2'(z_2)],\\
\sigma_{yy} &= 2\,\mathscr{Re}\,[\Phi_1'(z_1)+\Phi_2'(z_2)],\\
\sigma_{xy} &= -2\,\mathscr{Re}\,[\mu_1\Phi_1'(z_1)+\mu_2\Phi_2'(z_2)],\\
u_x &= 2\,\mathscr{Re}\,[p_1\Phi_1(z_1)+p_2\Phi_2(z_2)]-\omega y+u_x^0,\\
u_y &= 2\,\mathscr{Re}\,[q_1\Phi_1(z_1)+q_2\Phi_2(z_2)]+\omega x+u_y^0,
\end{aligned} \tag{2}$$

where p_k and q_k $(k = 1, 2)$ are given by

$$\begin{aligned}
p_k &= a_{11}\mu_k^2+a_{12}-a_{16}\mu_k,\\
q_k &= a_{12}\mu_k+a_{22}/\mu_k-a_{26},
\end{aligned} \tag{3}$$

and the prime denotes differentiation with respect to the corresponding argument. The rigid-body rotation and displacements are denoted by ω and u_x^0 and u_y^0, respectively. The components of resultant forces, $F_x(s)$ and $F_y(s)$, can be expressed in terms of the traction components X_n and Y_n, per unit thickness, acting along the $(s-s_0)$ arc of the boundary B_e and B_i of the laminate (Fig. 1):

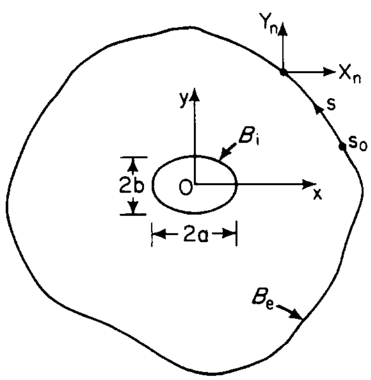

Fig. 1. Finite anisotropic laminate with a hole.

$$F_x(s) = \int_{s_0}^{s} X_n \, ds,$$

$$F_y(s) = \int_{s_0}^{s} Y_n \, ds. \tag{4}$$

The parameter s_0 indicates an arbitrary point (x_0, y_0) on the boundary, and s is an arc length parameter measured in the counterclockwise direction from s_0.

The components of the resultant forces, F_x and F_y, in terms of the analytic functions, become

$$\pm F_x(s) = 2\,\mathcal{Re}\,[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)] - 2\,\mathcal{Re}\,[\mu_1\Phi_1(z_1^0)+\mu_2\Phi_2(z_2^0)],$$
$$\mp F_y(s) = 2\,\mathcal{Re}\,[\Phi_1(z_1)+\Phi_2(z_2)] - 2\,\mathcal{Re}\,[\Phi_1(z_1^0)+\Phi_2(z_2^0)]. \tag{5}$$

Upper and lower signs refer to the exterior and interior boundaries, respectively, and $z_1^0 = x_0+\mu_1 y_0$ and $z_2^0 = x_0+\mu_2 y_0$. The second terms in eqns (5) correspond to the complex integration constants recognized by Bowie (1974) as the source of the inability to enforce displacement constraints directly. These terms (integration constants), which are dependent on the geometry and loading conditions, are retained in this formulation as part of the solution procedure. Determination of the stress distribution in an elastic anisotropic plate requires the explicit form of the analytic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ such that the specified boundary conditions are satisfied.

The solution procedure begins with the assumption that the analytic functions for $\Phi_1(\xi_1)$ and $\Phi_2(\xi_2)$ are in the form of infinite series:

$$\Phi_1(\xi_1) = \alpha_0 \ln \xi_1 + \sum_{n=1}^{\infty} (\alpha_{-n}\xi_1^{-n} + \alpha_n \xi_1^n),$$

$$\Phi_2(\xi_2) = \beta_0 \ln \xi_2 + \sum_{n=1}^{\infty} (\beta_{-n}\xi_2^{-n} + \beta_n \xi_2^n), \tag{6}$$

where α_i and β_i are unknown complex coefficients determined by enforcing the boundary conditions. As indicated by Lekhnitskii (1968), the logarithmic terms drop out if the force resultants on the interior boundary are zero. The series expansions for Φ_1 and Φ_2 are considered in mapped coordinates ξ_1 and ξ_2, rather than z_1 and z_2, in order to improve convergence. The mapping functions, ξ_1 and ξ_2, given by Lekhnitskii are employed:†

$$\xi_1 = \frac{z_1 \pm \sqrt{z_1^2 - a^2 - \mu_1^2 b^2}}{a - i\mu_1 b},$$

$$\xi_2 = \frac{z_2 \pm \sqrt{z_2^2 - a^2 - \mu_2^2 b^2}}{a - i\mu_2 b}, \tag{7}$$

where $i = \sqrt{-1}$ and a and b are the major and minor axes of an elliptical hole in a finite anisotropic laminate (Fig. 1).

In the series expansions for $\Phi_1(\xi_1)$ and $\Phi_2(\xi_2)$, the logarithmic terms are multi-valued; in order to ensure single-valued displacement components, the following conditions must be imposed:

† The sign of the square root terms in the mapping functions is chosen such that the internal boundary is mapped to unit circles. Note that these functions are analytic and hence the mappings are conformal.

$$\mathscr{I}m\,(p_1\alpha_0+p_2\beta_0) = 0 \quad \text{and} \quad \mathscr{I}m\,(q_1\alpha_0+q_2\beta_0) = 0. \tag{8}$$

It is implicit in the equations that the rigid-body translations, u_x^0 and u_y^0, are zero. The rigid-body rotation, ω, of an infinitesimal element located at (x,y) in the laminate must also vanish, i.e.

$$\omega = \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} = 0. \tag{9}$$

This results in the enforcement of the following condition:

$$\mathscr{R}e\left[\frac{\mu_1 p_1 - q_1}{a - i\mu_1 b}\alpha_1 + \frac{\mu_2 p_2 - q_2}{a - i\mu_2 b}\beta_1\right] = 0. \tag{10}$$

The solution is obtained by considering finite terms in the series as shown below:

$$\Phi_1(\xi_1) = \alpha_0 \ln \xi_1 + \sum_{n=1}^{N} (\alpha_{-n}\xi_1^{-n} + \alpha_n \xi_1^n),$$

$$\Phi_2(\xi_2) = \beta_0 \ln \xi_2 + \sum_{n=1}^{N} (\beta_{-n}\xi_2^{-n} + \beta_n \xi_2^n). \tag{11}$$

In order to determine the finite number $(4N+2)$ of unknown coefficients in the series given by eqn (11), M number of collocation points on the boundary are selected for imposing the boundary conditions.† The resulting overdetermined algebraic equations are solved by means of the lest squares procedure as introduced by Newmann (1971). Once the coefficients are determined, the stresses and displacements can be calculated using eqn (2).

3. NUMERICAL RESULTS

This study presents results for three different problems. The first problem (see insert in Fig. 3) involves a finite anisotropic laminate with a circular hole subjected to uniform tractions. The second and third problems (see inserts in Figs 5 and 6) involve a finite specially orthotropic laminate with two and four circular holes, respectively, under uniform tractions.‡ The circular holes present in each of these configurations are subjected to traction-free conditions; this is not due to any restrictions of the analysis presented herein.

To ensure the accuracy of the numerical results, the number of positive and negative terms, N, in the truncated series for $\Phi_1(\xi_1)$ and $\Phi_2(\xi_2)$ is chosen to be 10, and the numbers of collocation points on the exterior and interior boundaries are 44 and 52, respectively. These numerical values were established through a convergence study. The location and numbering of the collocation points are illustrated in Fig. 2.

Each lamina considered in this study is composed of homogeneous, elastic and orthotropic material with the following properties representative of graphite-epoxy: $E_L = 19.0\times10^6$ psi, $E_T = 1.9\times10^6$ psi, $G_{LT} = 0.9\times10^6$ psi, and $\nu_{LT} = 0.3$, where E_L and E_T are Young's moduli, G_{LT} is the shear modulus, ν_{LT} is Poisson's ratio, and subcripts L and T indicate the longitudinal and transverse directions relative to the fibers in the lamina. The results are independent of the thickness, t, of each lamina due to normalization of the compliance coefficients, a_{ij}, with respect to the actual laminate thickness.

† The number of collocation points, M, is usually much larger than the number of unknown coefficients in the series.

‡ In the case of specially orthotropic laminates, the principal axes of material symmetry coincide with the reference coordinate system.

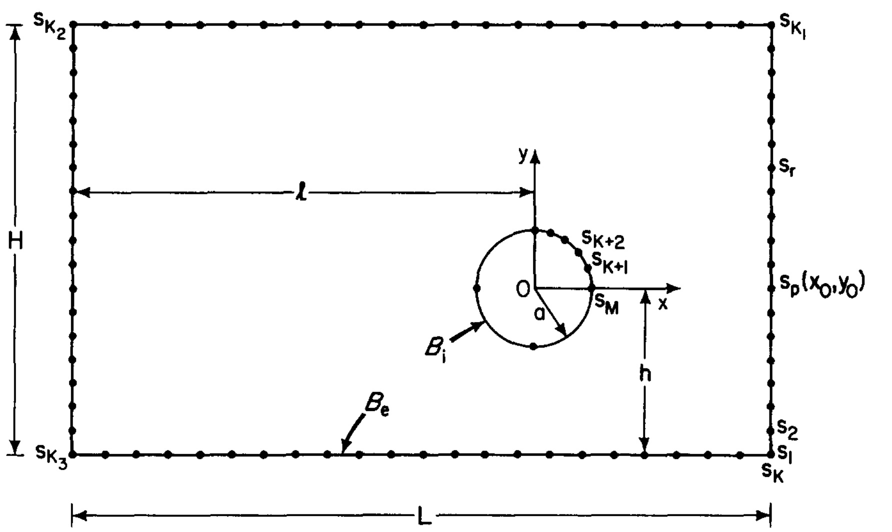

Fig. 2. Collocation points associated with the exterior and interior boundaries of a finite geometry laminate.

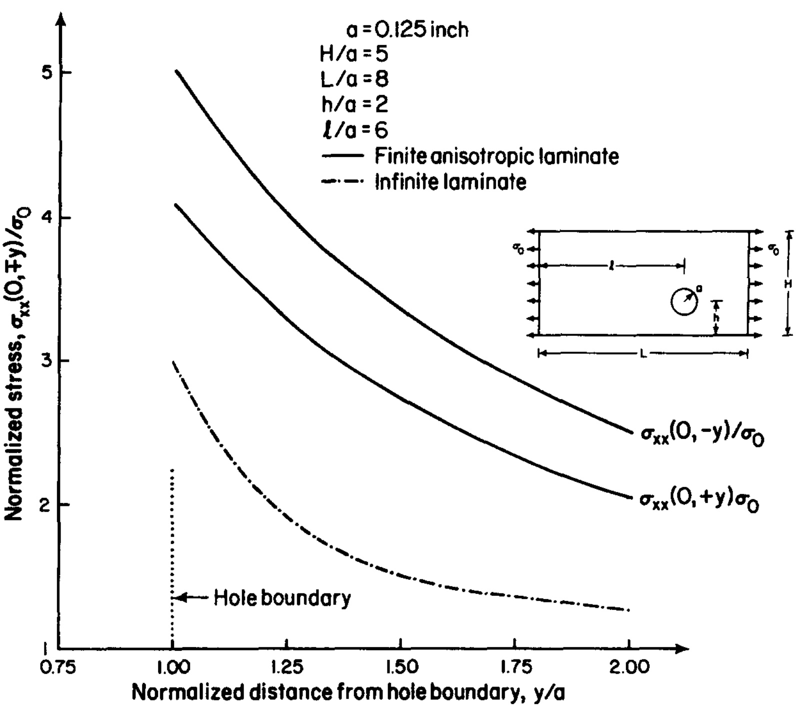

Fig. 3. Variation of normalized stress along y-axis in an anisotropic laminate with a circular hole.

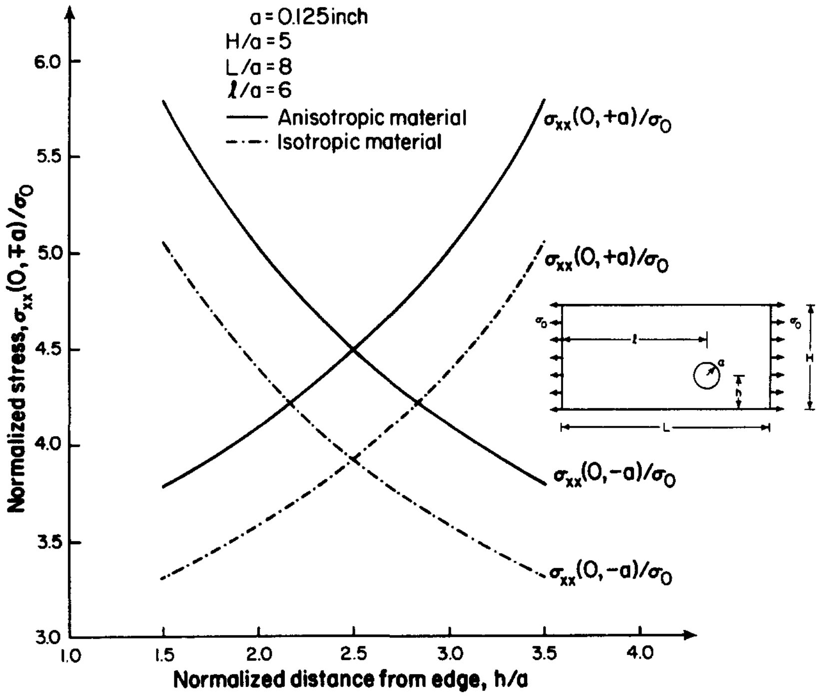

Fig. 4. Effect of laminate width on stress concentration.

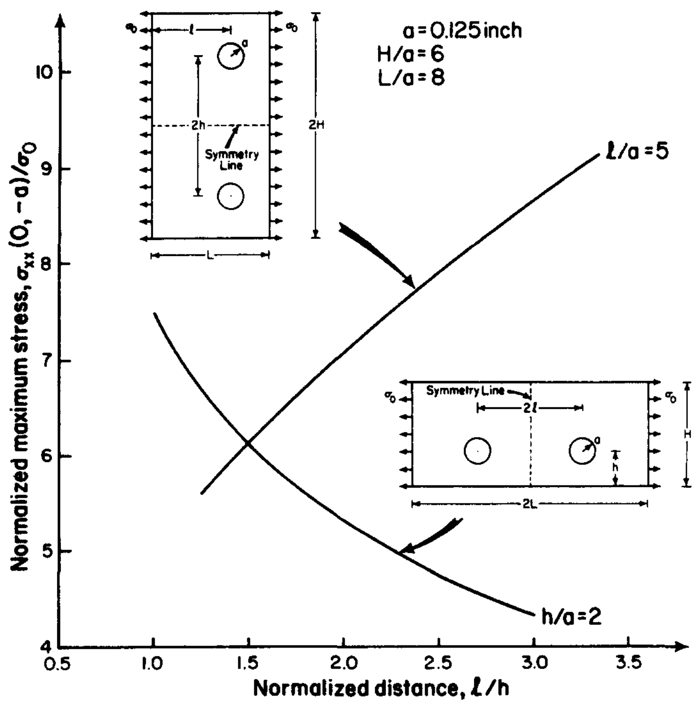

Fig. 5. Effect of distance between two holes and laminate width on stress concentration.

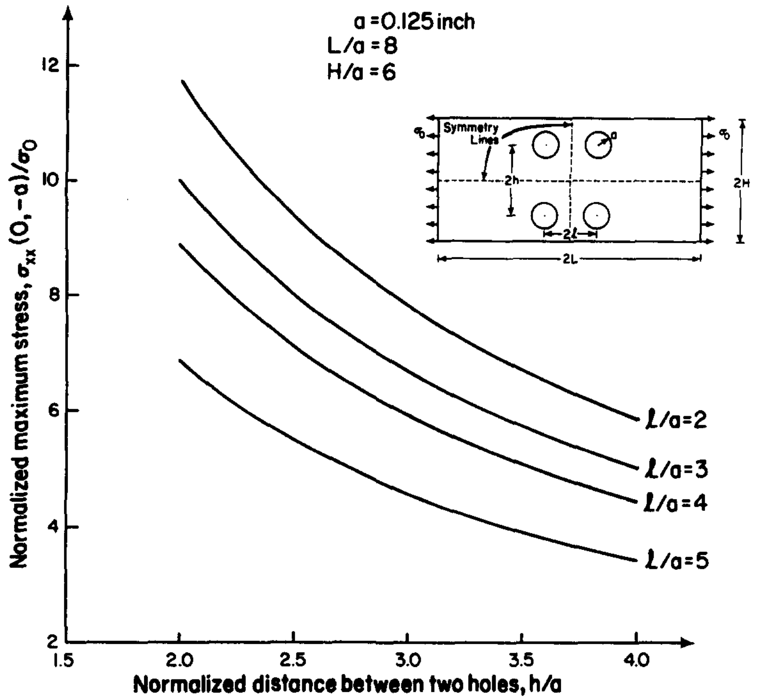

Fig. 6. Effect of distances between four holes on stress concentration.

Problem 1 : Anisotropic laminate with a circular hole

In this case, the laminate with layup $(0°_5/\pm 45_2/90°)_s$ is subjected to uniform tractions, σ_0. The boundary conditions associated with the collocation points that are shown in Fig. 2 can be expressed as:

Uniform tractions

$$
\begin{aligned}
F_x &= \sigma_0(s_r - y_0), & 1 \leqslant r \leqslant K_1, \\
F_x &= \sigma_0(H-h), & K_1 \leqslant r \leqslant K_2, \\
F_x &= \sigma_0(s_r - y_0), & K_2 \leqslant r \leqslant K_3, \\
F_x &= -\sigma_0 h, & K_3 \leqslant r \leqslant K \text{ and } r = 1, \\
-F_y &= 0, & 1 \leqslant r \leqslant K, \\
-F_x &= 0, & K+1 \leqslant r \leqslant M, \\
F_y &= 0, & K+1 \leqslant r \leqslant M,
\end{aligned}
$$

where s_r denotes the location of the collocation point, and the subscript r indicates the index of the collocation points on the exterior and interior boundaries. The geometric parameters h and H are illustrated in Fig. 2. The indices of the collocation points are chosen as $K_1 = 12$, $K_2 = 23$, $K_3 = 34$, $K = 44$ and $M = 96$.

The normalized maximum stress, $\sigma_{xx}(0, y)/\sigma_0$, is calculated for both finite and infinite anisotropic laminates.† These results are depicted in Fig. 3. The effect of finite boundaries on the stress concentration in anisotropic and isotropic materials is presented in Fig. 4. In

† A laminate with large dimensions is considered when the results are compared with the existing infinite plate solution.

Table 1. Normalized stress from a hole boundary in an infinite orthotropic laminate

y/a	$\sigma_{xx}(0,y)/\sigma_0$ Present analysis	Exact solution (Konish and Whitney, 1975)
1.0	6.871	6.863
1.2	2.078	2.070
1.4	1.534	1.530
1.6	1.311	1.310
1.8	1.163	1.160
2.0	1.123	1.120

the case of an infinite orthotropic laminate, the results tabulated in Table 1 from the present study and those of the closed-form solution given by Konish and Whitney (1975) almost coincide with each other. (The lamina properties of $E_L = 20.449 \times 10^6$ psi, $E_T = 1.3691 \times 10^6$ psi, $G_{LT} = 7.5216 \times 10^5$ psi, and $\nu_{LT} = 0.31$ were used to generate Table 1 for the purpose of comparison.)

Problem 2: Orthotropic laminate with two circular holes

Due to the presence of symmetry in geometry, material and loading, only half the laminate with layup ($0°_8$) is considered. As shown by the inserts in Fig. 5, configurations involving two holes either in series or parallel are investigated. For each case, the boundary conditions associated with the collocation points are specified as:

Two holes in series

$$
\begin{aligned}
F_x &= \sigma_0(s_r - y_0), && 1 \leqslant r \leqslant K_1, \\
F_x &= \sigma_0(H-h), && K_1 \leqslant r \leqslant K_2, \\
u_x &= 0, && K_2 \leqslant r \leqslant K_3, \\
F_x &= -\sigma_0 h, && K_3 \leqslant r \leqslant K \text{ and } r = 1, \\
-F_y &= 0, && 1 \leqslant r \leqslant K, \\
-F_x &= 0, && K+1 \leqslant r \leqslant M, \\
F_y &= 0, && K+1 \leqslant r \leqslant M.
\end{aligned}
$$

Table 2. Normalized maximum stresses in an infinite quasi-isotropic laminate

a. With two holes in series

l/a	$\sigma_{xx}(0, \mp a)/\sigma_0$ Present analysis	Peterson (1974)
1.5	2.629	2.623
2.0	2.705	2.703
3.0	2.822	2.825
5.0	2.930	2.927
8.0	2.972	2.970

b. With two holes in parallel

h/a	$\sigma_{xx}(0, -a)/\sigma_0$ Present analysis	Peterson (1974)
1.5	3.267	3.264
2.0	3.061	3.060
3.0	3.002	3.002
5.0	2.999	2.999
8.0	2.999	2.999

Table 3. Normalized maximum stresses in an infinite quasi-isotropic laminate with four holes

$h = l$	$\sigma_{xx}(0, -a)/\sigma_0$	
$(l-a)/l$	Present analysis	Peterson (1974)
0.15	7.40	7.45
0.30	4.25	4.26
0.40	3.50	3.55
0.70	3.05	3.05
0.90	3.00	3.00
1.00	3.00	3.00

Two holes in parallel

$$
\begin{aligned}
F_x &= \sigma_0(s_r - y_0), && 1 \leqslant r \leqslant K_1,\\
F_x &= \sigma_0(H-h), && K_1 \leqslant r \leqslant K_2,\\
F_x &= \sigma_0(s_r - y_0), && K_2 \leqslant r \leqslant K_3,\\
-F_y &= 0, && 1 \leqslant r \leqslant K_3,\\
u_y &= 0, && K_3 \leqslant r \leqslant K \text{ and } r = 1,\\
F_x &= -\sigma_0 h, && K_3 \leqslant r \leqslant K \text{ and } r = 1,\\
-F_x &= 0, && K+1 \leqslant r \leqslant M,\\
F_y &= 0, && K+1 \leqslant r \leqslant M.
\end{aligned}
$$

The variation of normalized maximum stress, $\sigma_{xx}(0, \pm a)/\sigma_0$, for these configurations is presented in Fig. 5. The infinite quasi-isotropic laminate with the same geometric configurations is also considered in order to establish the validity of the present analysis results. As tabulated in Table 2, these results agree with those given by Peterson (1974) for an infinite quasi-isotropic plate.

Problem 3 : Orthotropic laminate with four circular holes

The geometry and loading conditions of this case are illustrated in Fig. 6. Since the laminate is specially orthotropic with layup $(0°_8)$, it is sufficient to consider only a quarter of the laminate. The boundary conditions imposed on the collocation points are:

$$
\begin{aligned}
F_x &= \sigma_0(s_r - y_0), && 1 \leqslant r \leqslant K_1,\\
F_x &= \sigma_0(H-h), && K_1 \leqslant r \leqslant K_2,\\
u_x &= 0, && K_2 \leqslant r \leqslant K_3,\\
u_y &= 0, && K_3 \leqslant r \leqslant K \text{ and } r = 1,\\
-F_y &= 0, && 1 \leqslant r \leqslant K,\\
-F_x &= 0, && K+1 \leqslant r \leqslant M,\\
F_y &= 0, && K+1 \leqslant r \leqslant M.
\end{aligned}
$$

The normalized maximum stresses, $\sigma_{xx}(0, -a)/\sigma_0$, are calculated for an infinite quasi-isotropic laminate. These results and their comparison to those given by Peterson (1974) are presented in Table 3. The effect of finite geometry and the position of the holes on the stress concentration is shown in Fig. 6.

4. CONCLUSIONS

The present analysis focuses on the assessment of the stress concentration in finite composite laminates with holes. The problems considered in this study were chosen so as to demonstrate the accuracy and versatility of the MMC method. As observed in Tables 1–3, the results compare favorably with known solutions. The results also illustrate the effect of finite boundaries and their relative position in relation to the holes on the stress concentrations.

A significant contribution of this study is that it allows for tractions, displacement constraints, and mixed boundary conditions. This approach can be applied to other problems with more complex geometric and loading conditions. For example, it may be well suited for the determination of stress distributions around a hole with a fastener in a finite composite laminate.

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